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New exact solutions for inflationary cosmological perturbations

Jérôme Martin[a1] & Dominik J. Schwarz[b2]

[a] *DARC, Observatoire de Paris-Meudon, UMR 8629 CNRS, 92195 Meudon Cedex, France.*
[b] *Institut für Theoretische Physik, Technische Universität Wien, Wiedner Hauptstraße 8–10, 1040 Wien, Austria.*

Abstract

From a general ansatz for the effective potential of cosmological perturbations we find new, exact solutions in single-scalar-field inflation: A three parameter family of exact solutions that encompasses all exact solutions that have been known previously (power-law inflation, Easther's model, and a generalized version of Starobinsky's solution). The main feature of this new family is that the spectral indices are scale dependent.

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[1]email: martin@edelweiss.obspm.fr
[2]email: dschwarz@hep.itp.tuwien.ac.at

1 Introduction

The observation of the first acoustic peak in the spectrum of anisotropies in the CMBR (Cosmic Microwave Background Radiation) [1, 2] leaves inflation as the only mechanism that can provide seeds for the observed large-scale structure of the Universe. The theory of inflation predicts primordial density perturbations [3] and primordial gravitational waves [4] from quantum fluctuations of the vacuum during the inflationary epoch. So far, most predictions of inflationary cosmology have been based either on the slow-roll approximation [5] or on power-law inflation [6]. However, it is clear that inflation might occur in a much broader context and we should discriminate between generic predictions of inflation and predictions of a specific scenario.

The generic features of inflation might be studied from two sides: from the particle physics point of view the starting point is the inflaton potential $V(\varphi)$, together with initial conditions for φ and $\dot{\varphi}$. From the cosmological side, one might start from the effective potential (or effective action) for the evolution of cosmological perturbations. Here we choose the latter approach. We find a family of exact solutions for the mode equations of cosmological perturbations that includes all previously known solutions as limiting cases and that allows us to gain new insight into the generic properties of inflation. Especially, we find primordial spectra with k-dependent spectral indices and that are non-analytic in the wavenumber k.

Let us now introduce some basic tools for the study of cosmological perturbations. During inflation, density (scalar) perturbations and gravitational waves (tensor perturbations) can be characterized by the quantities $\mu_{\mathrm{S,T}}(\eta)$ [7], which obey the equations (we use the notation of [8, 9]):

$$\mu_{\mathrm{S,T}}'' + \left[k^2 - \frac{z_{\mathrm{S,T}}''}{z_{\mathrm{S,T}}}\right]\mu_{\mathrm{S,T}} = 0, \tag{1}$$

where a prime denotes differentiation with respect to conformal time η and $z_{\mathrm{S,T}}$ are functions of the scale factor $a(\eta)$ and of its derivatives only:

$$z_{\mathrm{S}}(\eta) \equiv a\sqrt{\gamma}, \quad z_{\mathrm{T}}(\eta) \equiv a, \tag{2}$$

with $\gamma(\eta) \equiv 1 - (a^2/a'^2)(a'/a)'$. Eq. (1) may be viewed as a "time-independent" Schrödinger equation with an effective potential given by

$$U_{\mathrm{S,T}}(\eta) \equiv \frac{z_{\mathrm{S,T}}''}{z_{\mathrm{S,T}}}. \tag{3}$$

The assumption that the quantum fields are initially placed in the vacuum state, when the mode k is subhorizon, fixes the initial conditions for the quantities $\mu_{\mathrm{S,T}}$. They read:

$$\lim_{k/(aH)\to+\infty} \mu_{\mathrm{S,T}}(\eta) = \mp 4\sqrt{\pi}\, l_{\mathrm{Pl}} \frac{e^{-ik(\eta-\eta_{\mathrm{i}})}}{\sqrt{2k}}, \tag{4}$$

where H is the Hubble parameter and η_{i} an arbitrary initial time at the beginning of inflation. Then, an integration of Eq. (1) allows the determination of the power spectra $P(k)$. For density perturbations we choose to work with the power spectrum of the hypersurface independent quantity ζ, which corresponds to the perturbation of the intrinsic curvature of a spatial, uniform density hypersurface. For gravitational waves we determine the power spectrum of the amplitude h. These power spectra are calculated according to:

$$k^3 P_\zeta \equiv \frac{k^3}{8\pi^2}\left|\frac{\mu_{\mathrm{S}}}{z_{\mathrm{S}}}\right|^2, \quad k^3 P_h \equiv \frac{2k^3}{\pi^2}\left|\frac{\mu_{\mathrm{T}}}{z_{\mathrm{T}}}\right|^2. \tag{5}$$

Measurements of the CMBR anisotropies and of the large-scale structure probe scales which were well beyond the Hubble radius at the end of inflation. Thus, we are interested in the modes which satisfy $k/(aH) \ll 1$ at the end of inflation. It is easy to see from Eq. (1) that the spectra are time independent in this regime since $\mu_{\mathrm{S,T}} \propto z_{\mathrm{S,T}}$ once the subdominant mode can be neglected. Let us also note that the spectrum of the quantity ζ is related to the spectrum of the Bardeen potential today by $k^3 P_\zeta = (25/9) k^3 P_\Phi$. The spectral indices are defined by: $n_{\mathrm{S}} - 1 \equiv \mathrm{d}\ln(k^3 P_\zeta)/\mathrm{d}\ln k$ and $n_{\mathrm{T}} \equiv \mathrm{d}\ln(k^3 P_h)/\mathrm{d}\ln k$.

The behaviour of cosmological perturbations during inflation is completely fixed once the functions $z_{\mathrm{S,T}}(\eta)$ are specified. The effective potential $U_{\mathrm{S,T}}(\eta)$ follows from Eq. (3) and Eq. (1) can be solved. Integration of equation (2) provides the scale factor. Instead of assuming specific functions $z_{\mathrm{S,T}}(\eta)$, one can also start from the effective potential $U_{\mathrm{S,T}}(\eta)$ itself, such that Eq. (1) may be solved analytically. In this case the functions $z_{\mathrm{S,T}}(\eta)$ are known explicitly since $z_{\mathrm{S,T}}(\eta) = \mu_{\mathrm{S,T}}(k=0,\eta)$.

Let us now discuss the shape of $U_{\mathrm{S,T}}(\eta)$. One expects that the effective potential possesses the following properties: $\lim_{|\eta|\to+\infty} U_{\mathrm{S,T}}(\eta) \ll k^2$, for any wavenumber of interest, in order for the modes to be inside the horizon at the initial time η_{i} (so that we can fix the normalization from quantum-mechanical considerations); it seems also reasonable to assume that $\lim_{|\eta|\to 0} U_{\mathrm{S,T}}(\eta) \gg k^2$ which guarantees that the quantum fluctuations are amplified (i.e. are frozen outside the horizon). These requirements do not single out a unique effective potential. However, a general simple ansatz satisfying these conditions is given by:

$$U_{\mathrm{S,T}}(\eta) = \sum_{m=1}^{M} \frac{c_m}{|\eta|^m}, \tag{6}$$

where the c_m's should be determined for each specific model of inflation. It is necessary to have $c_{\min}$ and c_M positive. The coefficient c_1 defines a characteristic scale $k_{\mathrm{C}} \approx c_1$. The corresponding term in the series (6) dominates the other ones when $|\eta| \to \infty$ and will therefore determine the power spectrum at very large scales.

Inflationary models for which the evolution of cosmological perturbations has been studied so far, power-law inflation and slow-roll inflation, are chosen such that $c_m = 0$ if $m \neq 2$. In this sense they are special models. In particular, they lead to k-independent spectral indices.

In this article we study the behaviour of cosmological perturbations for models characterized by an effective potential given in Eq. (6). A first step in the analysis of these models consists in studying the effective potential

$$U_{\mathrm{S,T}}(\eta) = \frac{c_1}{|\eta|} + \frac{c_2}{|\eta|^2}. \tag{7}$$

The final power spectra depend on the free parameters c_1 and c_2 plus one parameter contained in $z_{\mathrm{S,T}}(\eta)$. We demonstrate that the general solution for this potential is given in terms of Whittaker functions and we show explicitly how all the previously known cases can be recovered from this more general solution. We prove that the power spectra are such that the spectral indices depend on k.

Although all the properties of the cosmological perturbations can be calculated analytically, it is not possible to determine the corresponding inflaton potential analytically. This issue is important since we would like to know whether the new class of solutions corresponds to generic potentials $V(\varphi)$ from the particle physics point of view. We calculate the corresponding inflaton potentials numerically for the scalar case, with initial conditions inspired by chaotic inflation.

This letter is organized as follows. In section II, we briefly remind the reader about known analytical solutions. In section III, we present a new family of exact solutions and in section IV we study various limits of the parameters of the new solutions.

2 Known analytic solutions

In the simplest case all coefficients c_m vanish and $U_{\mathrm{S,T}} = 0$ [10]. The corresponding functions $z_{\mathrm{S,T}}$ are given by $z_{\mathrm{S,T}}(\eta) \equiv B\eta + A$, where A and B are free parameters. For the tensor sector, this gives just the radiation dominated Universe. For the scalar sector, the general solution of Eq. (1) can be written as $\mu_{\mathrm{S}}(\eta) = C_1 e^{ik\eta} + C_2 e^{-ik\eta}$, where C_1 and C_2 are two arbitrary constants to be determined by the initial conditions (4). They read $C_1 = 0$ and $C_2 = -4\sqrt{\pi} l_{\mathrm{Pl}} e^{ik\eta_{\mathrm{i}}}/\sqrt{2k}$. In the large scale limit we have $z_{\mathrm{S}} \approx A$ and the spectrum of density perturbations reads:

$$k^3 P_\zeta = \frac{l_{\mathrm{Pl}}^2}{\pi A^2} k^2. \tag{8}$$

The spectral index is given by $n_{\mathrm{S}} = 3$, which is in obvious contradiction with observations. The analytic form of the corresponding scalar potential $V(\varphi)$ has been given in Ref. [10].

All other cases known so far assume a potential such that $U_{\mathrm{S,T}} = \alpha(\alpha+1)/\eta^2$, i.e. $c_m = 0$ if $m \neq 2$ and $c_2 = \alpha(\alpha+1)$, where α is a free parameter [9]. The corresponding functions $z_{\mathrm{S,T}}$ can be expressed as $z_{\mathrm{S,T}}(\eta) = A/|\eta|^\alpha$, where A is another free parameter. Thus, we have a two-parameter family characterized by A and α. For tensor perturbations all these models correspond to power-law inflation [6]. For scalar perturbations the subclass $\alpha = 1$ was studied in Ref. [11]. Another subclass is $A = l_0\sqrt{(\alpha-1)/\alpha}$, which corresponds to power-law inflation [6] with the scale factor given by $a(\eta) = l_0|\eta|^{-\alpha}$. The quantity l_0 has the dimension of a length. In this model inflation occurs if $\alpha \geq 1$. In the de Sitter case ($\alpha = 1$), l_0 is simply the constant Hubble radius during inflation. Only for power-law inflation scalar and tensor perturbations can be solved analytically at the same time. For scalar perturbations the general two-parameter family is not equivalent to power-law inflation. Of course, $a(\eta) \propto |\eta|^{-\alpha}$ is a particular solution of $z_{\mathrm{S}}(\eta) = A/|\eta|^\alpha$, but it is not the general solution. The limit A to zero and α close to one reproduces a slow-roll inflation model. Eq. (1) is solved in terms of Bessel functions, $\mu_{\mathrm{S}}(\eta) = (k\eta)^{\frac{1}{2}}[C_1 J_{\alpha+\frac{1}{2}}(k\eta) + C_2 J_{-\alpha-\frac{1}{2}}(k\eta)]$, where and C_1 and C_2 are constants fixed from Eq. (4). The spectrum may be calculated exactly to read

$$k^3 P_\zeta(k) = \frac{l_{\mathrm{Pl}}^2}{\pi^2 A^2} 2^{2\alpha}\Gamma^2\left(\alpha + \frac{1}{2}\right) k^{-2(\alpha-1)}\ . \tag{9}$$

The corresponding spectral index is $n_{\mathrm{S}} = 3 - 2\alpha$. In particular, we have $k^3 P_\zeta = l_{\mathrm{Pl}}^2/(\pi A^2)$ and $n_{\mathrm{S}} = 1$ for $\alpha = 1$. For gravitational waves the same ansatz gives a power-law model with spectral index $n_{\mathrm{T}} = 2 - 2\alpha$.

The inflaton potential for the scalar sector is displayed in Fig. (1) for various values of α. We use initial conditions motivated by the scenario of chaotic inflation, i.e. the potential and the kinetic energy densities of the inflaton field are of the order of the Planck energy density initially. For power-law inflation, the potential is given by: $V(\varphi) = V_{\mathrm{i}} \exp[4\sqrt{\pi\gamma}(\varphi - \varphi_{\mathrm{i}})/m_{\mathrm{Pl}}]$.

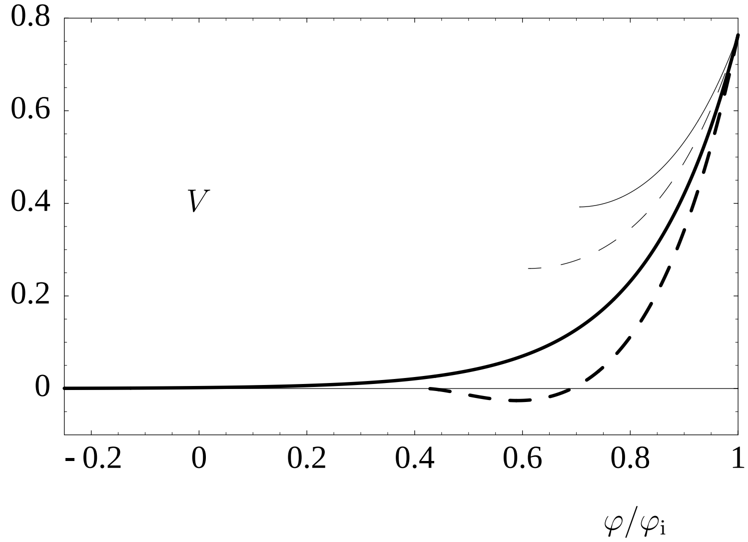

Figure 1: The generalized Starobinsky potentials for $\alpha = 1, 2, \alpha_\mathrm{pl} \equiv -\beta - 1, 5$ (from top to bottom) and $A = 1$, where $\alpha_\mathrm{pl} \approx 3.44$. The initial conditions are $\gamma_\mathrm{i} = (\alpha_\mathrm{pl} - 1)/\alpha_\mathrm{pl} \approx 0.7$ and $H_\mathrm{i} = \sqrt{8\pi G/3}$.

3 A new analytical solution

We now turn to the main result of this article. The function $z_\mathrm{S,T}$ corresponding to the effective potential (7) can be written as :

$$z_\mathrm{S,T}(\eta) = \frac{2Ac_1^{\xi-1/2}}{\Gamma(2\xi)}\sqrt{c_1|\eta|}K_{2\xi}(2\sqrt{c_1|\eta|}) \ , \tag{10}$$

where $A > 0$ and $\xi \equiv \sqrt{c_2 + 1/4} \geq 1/2$. $K_{2\xi}$ is a modified Bessel function of order 2ξ. We now define τ as $\tau \equiv 2ik\eta$ and κ as $\kappa \equiv -ic_1/(2k)$. c_1 defines a characteristic scale given by $k_\mathrm{C} = c_1/2$. Then, the equation of motion (1) can be expressed as:

$$\frac{\mathrm{d}^2\mu_\mathrm{S,T}}{\mathrm{d}\tau^2} + \left[-\frac{1}{4} + \frac{\kappa}{\tau} + \frac{(1/4 - \xi^2)}{\tau^2}\right]\mu_\mathrm{S,T} = 0. \tag{11}$$

The general solution to this equation is given in terms of Whittaker functions

$$\mu_\mathrm{S,T}(\tau) = C_1 W_{\kappa,\xi}(\tau) + C_2 W_{-\kappa,\xi}(-\tau). \tag{12}$$

The constants C_1 and C_2 are fixed by the initial conditions. Using the asymptotic behaviour of the Whittaker functions in the small-scale limit, $\lim_{|\tau|\to\infty} W_{\kappa,\xi}(\tau) = e^{-\frac{\tau}{2}}\tau^\kappa$, Eq. (9.227) of Ref. [12], we find:

$$C_1 = \mp 4\sqrt{\pi}l_\mathrm{Pl}\frac{e^{ik\eta_\mathrm{i}+\pi c_1/(4k)}}{\sqrt{2k}}, \quad C_2 = 0 \ . \tag{13}$$

Expressing the Whittaker functions in terms of Kummer functions, Eqs. (9.220) of Ref. [12], we deduce that, if $-\xi + 1/2 < 0$, the growing mode in the large scale limit is given by: $\lim_{|\tau|\to 0} W_{\kappa,\xi}(\tau) = \Gamma(2\xi)/[\Gamma(1/2 + \xi - \kappa)](\tau)^{-\xi+1/2}e^{-\tau/2}$. Using that $z_\mathrm{S,T}(\eta) \approx A|\eta|^{\frac{1}{2}-\xi}$ in this limit, it is straightforward to calculate the spectrum of density perturbations:

$$k^3 P_\zeta = \frac{l_{\rm Pl}^2}{\pi A^2}\frac{\Gamma^2(2\xi)}{2^{2\xi-1}}\frac{k^{3-2\xi}e^{\pi k_{\rm C}/k}}{|\Gamma(1/2+\xi+ik_{\rm C}/k)|^2}. \tag{14}$$

This spectrum corresponds to a new exact solution which contains all the previously known cases as particular cases, see the next section. Using Eq. (8.328.1) of Ref. [12], we see that for $k \ll k_{\rm C}$ one has $|\Gamma(1/2+\xi+ik_{\rm C}/k)|^2 \approx 2\pi \exp(-\pi k_{\rm C}/k)(k_{\rm C}/k)^{2\xi}$. This implies that in the large-scale limit the spectrum is given by:

$$k^3 P_\zeta \approx \frac{l_{\rm Pl}^2 k^3}{\pi^2 A^2}\frac{\Gamma^2(2\xi)}{(2k_{\rm C})^{2\xi}}e^{2\pi k_{\rm C}/k}, \tag{15}$$

if $c_1 \neq 0$ and $c_0 = 0$. We see that it is not analytic in the region of small k. Of course, such an infra-red divergence of the spectrum is excluded from observation. However, the validity of our description fails for modes that leave the horizon at the Planck epoch, i.e. at the beginning of inflation in the scenario of chaotic inflation. This sets a natural infra-red cut-off to the power spectrum (14). In the small-scale limit $k \gg k_{\rm C}$ the spectrum tends to

$$k^3 P_\zeta \approx \frac{l_{\rm Pl}^2}{\pi^2 A^2} 2^{2\xi-1}\Gamma^2(\xi) k^{3-2\xi} \ . \tag{16}$$

This is the spectrum of the two parameter family studied in the previous section, see Eq. (9). The corresponding constant spectral index is given by $n_{\rm S} = 4 - 2\xi = 4 - 2\sqrt{c_2 + 1/4}$ and is entirely determined by the coefficient c_2.

4 Limiting cases

4.1 $c_1 \neq 0$, $c_2 = 0$: Coulomb solution

As already mentioned, the case usually treated in the literature is $c_1 = 0$, $c_2 \neq 0$. On the other hand, the case $c_1 \neq 0$, $c_2 = 0$ has never been studied before. Although it is of course just a particular case of the general solution given in the previous section, it is worth investigating its properties in some details. For this purpose, we restart from the beginning and consider the following function $z_{\rm S,T}$

$$z_{\rm S,T}(\eta) \equiv 2A\sqrt{c_1|\eta|}K_1\left(2\sqrt{c_1|\eta|}\right). \tag{17}$$

The link with Eq. (10) is obvious since $c_2 = 0$ corresponds to $\xi = 1/2$. Let us now define ρ and δ according to $\rho \equiv i\tau/2$, $\delta \equiv i\kappa$. Then the equation of motion (1) for $\mu_{\rm S,T}(\eta)$ can be written as:

$$\frac{{\rm d}^2\mu_{\rm S,T}}{{\rm d}\rho^2} + \left(1 - \frac{2\delta}{\rho}\right)\mu_{\rm S,T} = 0. \tag{18}$$

This equation can be solved in terms of Coulomb functions with $l = 0$, $\mu_{\rm S,T}(\rho) = C_1 F_0(\delta;\rho) + C_2 G_0(\delta;\rho)$. The coefficients C_1 and C_2 are fixed by the initial conditions, see Eq. (4). We find that $C_1 = \pm 4i\sqrt{\pi} l_{\rm Pl} e^{ik\eta_{\rm i}}/\sqrt{2k}$, $C_2 = -iC_1$. In the large scale limit, the growing mode is given by the irregular Coulomb wave function $G_0(\delta;\rho) \approx 1/C_0(\delta)$, where $C_0(\delta) \equiv \sqrt{2\pi\delta/(e^{2\pi\delta}-1)}$.

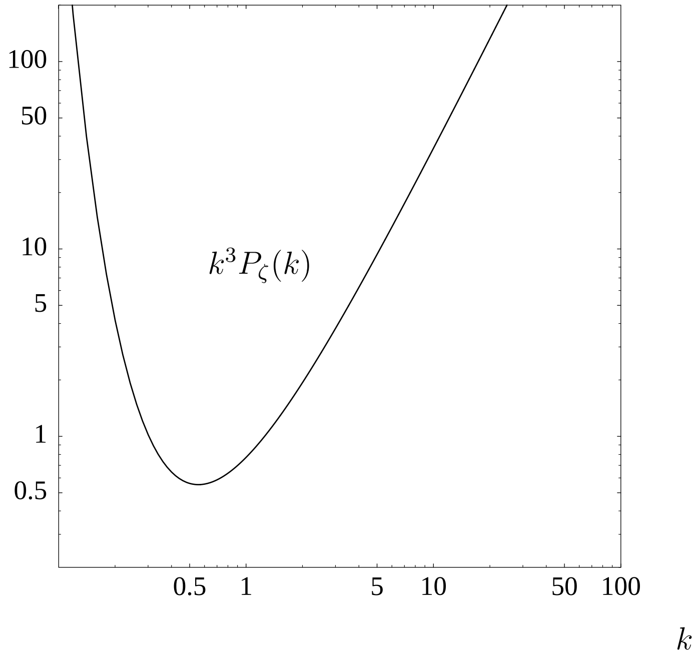

Figure 2: The scalar power spectrum for the Coulomb case with $A = 1$ and $c_1 = 0.5$.

Using the fact that $K_1(x) \approx 1/x$ when x goes to zero, the spectrum is easily derived. For density perturbations, we find:

$$k^3 P_\zeta = \frac{l_{\rm Pl}^2 k^3}{2\pi^2 A^2 k_{\rm C}} \left(e^{2\pi k_{\rm C}/k} - 1\right). \tag{19}$$

This spectrum is of course nothing but Eq. (14) for $\xi = 1/2$. It is displayed in Fig. (2) for $A = 1$ and $c_1 = 0.5$. In the regime where $k \ll k_{\rm C}$ one recovers Eq. (15) whereas if $k \gg k_{\rm C}$ this spectrum tends to the Easther's solution $k^3 P_\zeta = (l_{\rm Pl}^2 k^2)/(\pi A^2)$ for which $n_{\rm S} = 3$ in accordance with Eq. (16). The corresponding inflaton potentials for various values of c_1 are displayed in Fig. (3).

4.2 $c_1 = 0$, $c_2 \neq 0$: generalized power-law solution

Putting $c_1 = 0$ and $\kappa = 0$ in Eq. (14), one easily checks that the result is the generalized power-law spectrum, see Eq. (9), with $\alpha = \xi - 1/2$. The corresponding spectral index is $n_{\rm S} = 4 - 2\xi$.

4.3 $c_1 \neq 0$, $c_2 = 2$

Let us finally investigate in more details the case $c_2 = 2$ which corresponds to $\xi = 3/2$. Using Eq. (8.332.1) of Ref. [12], we find that the exact spectrum for density perturbations is given by:

$$k^3 P_\zeta^{(c_2=2)} = \frac{l_{\rm Pl}^2}{2\pi^2 A^2} \frac{k}{k_{\rm C}} \left(1 + \frac{k_{\rm C}^2}{k^2}\right)^{-1} \left(e^{2\pi k_{\rm C}/k} - 1\right). \tag{20}$$

The spectrum is displayed in Fig. (4) for $A = 1$ and $c_1 = 0.5$.

In the limit $k_{\rm C}/k \ll 1$, the previous spectrum reduces to $k^3 P_\zeta^{(c_2=2)} = (l_{\rm Pl}^2)/(\pi A^2)$, i.e. the spectrum of Starobinsky's solution ($n_{\rm S} = 1$) as expected. The potential of the inflaton is displayed in Fig. (5) for various values of c_1.

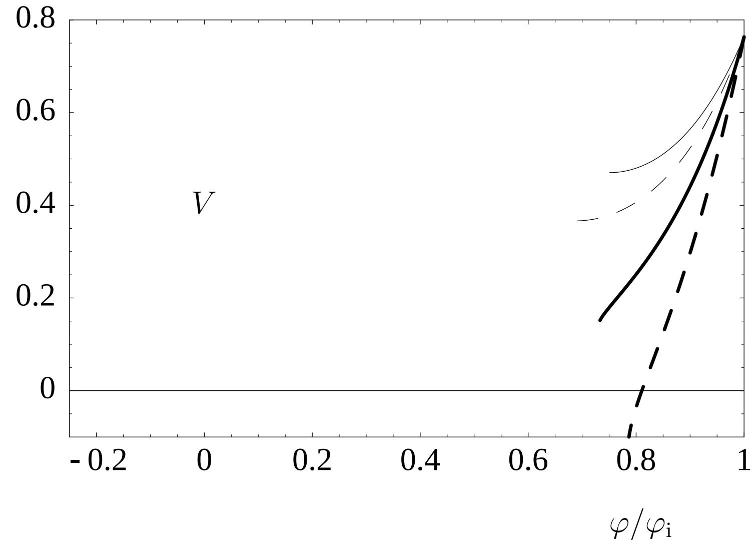

Figure 3: The potentials for the Coulomb case for $c_1 = 0.1, 0.5, 1, 1.5$ and $A = 1$ with the same initial conditions as before.

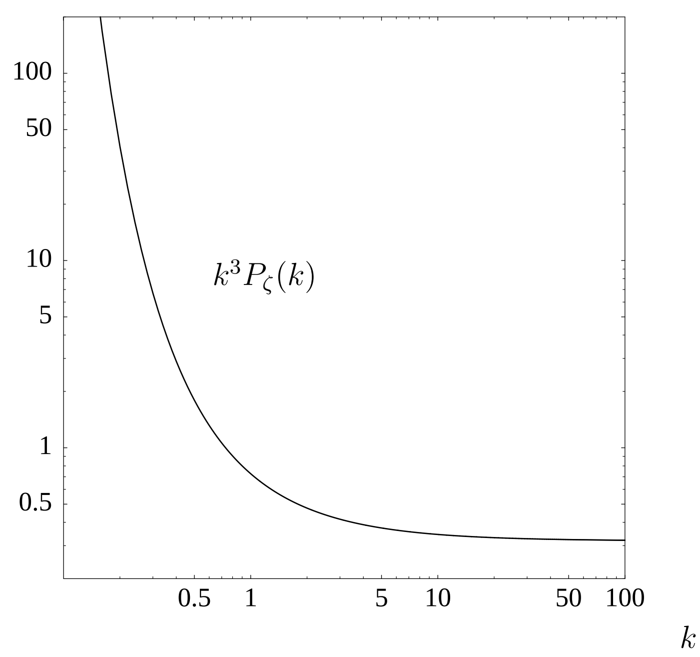

Figure 4: The scalar power spectrum for the Whittaker case with $A = 1, c_2 = 2$ and $c_1 = 0.5$.

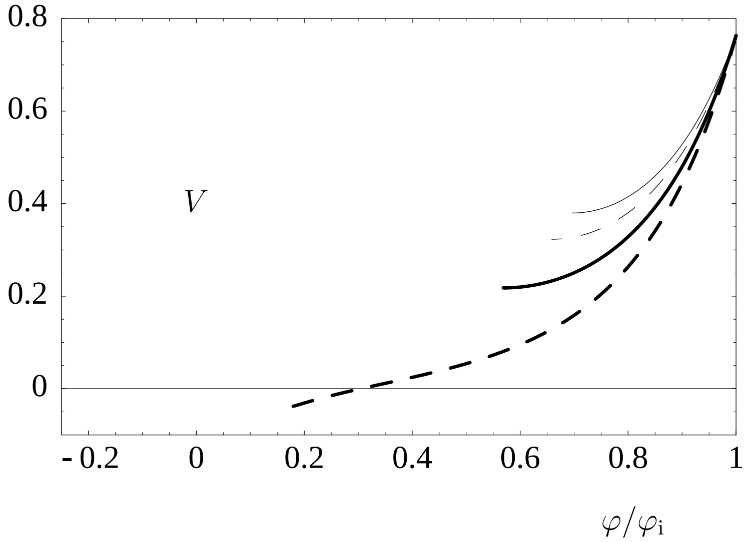

Figure 5: The potentials for the Whittaker case for $A = 1, c_2 = 2$ and $c_1 = 0.1, 0.5, 1, 1.5$ with the same initial conditions as before.

We briefly conclude in recalling that the main result of this article is the discovery of a new family of exact solutions for cosmological perturbations which encompasses all the previously known cases as limiting cases, see Table I. This new family could help to shed light on inflationary models which do not fulfill the slow-roll conditions and may be used to test approximate methods valid for models which possess scale-dependent spectral indices. In this spirit, the precision of the slow-roll approximation was recently investigated using power-law inflation as an exact solution [9].

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c_1	c_2	Regime	$k^3 P_\zeta$
$\neq 0$	$\neq 0$	Exact	Whittaker's solution (14), $n_S = n_S(k)$
$\neq 0$	$\neq 0$	$k \ll k_C$	Eq. (15), $n_S = n_S(k)$
$\neq 0$	$\neq 0$	$k \gg k_C$	Generalized power-law (9), $n_S = 4 - 2\sqrt{c_2 + 1/2}$
0	0	Exact	Easther's solution (8), $n_S = 3$
0	$\neq 0$	Exact	Generalized power-law (9), $n_S = 4 - 2\sqrt{c_2 + 1/2}$
$\neq 0$	0	Exact	Coulomb's solution (19), $n_S = n_S(k)$
$\neq 0$	0	$k \gg k_C$	Easther's solution (8), $n_S = 3$
$\neq 0$	2	Exact	Eq. (20), $n_S = n_S(k)$
$\neq 0$	2	$k \gg k_C$	Starobinski's solution (9), $n_S = 1$

Table 1: Spectra for different values of c_1 and c_2 and and in different regimes.

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